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Computing global offensive alliances in Cartesian product graphs

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ABSTRACT

A global offensive alliance in a graph G is a set S of vertices with the property that every vertex not belonging to S has at least one more neighbor in S than it has outside of S . The global offensive alliance number of G , $\gamma_o(G)$, is the minimum cardinality of a global offensive alliance in G . A set S of vertices of a graph G is a dominating set for G if every vertex not belonging to S has at least one neighbor in S . The domination number of G , $\gamma(G)$, is the minimum cardinality of a dominating set of G . In this work we obtain closed formulas for the global offensive alliance number of several families of Cartesian product graphs, we also prove that $\gamma_o(G \square H) \geq \frac{\gamma(G)\gamma_o(H)}{2}$ for any graphs G and H and we show that if G has an efficient dominating set, then $\gamma_o(G \square H) \geq \gamma(G)\gamma_o(H)$. Moreover, we present a Vizing-like conjecture for the global offensive alliance number and we prove it for several families of graphs.

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1. Introduction

Alliances in graphs were described first by Kristiansen et al. in [11], where alliances were classified into defensive, offensive or powerful. After this seminal paper, the issue has been studied intensively. Remarkable examples are the articles [13,14], where alliances were generalized to k -alliances, and [4], where the authors presented the first results on offensive alliances. One of the main motivations of this study is based on the NP-completeness of computing minimum cardinality of (defensive, offensive, powerful) alliances in graphs.

On the other hand, several graphs may be constructible from smaller and simpler components by basic operations like unions, joins, compositions, or multiplications with respect to various products, where properties of the constituents determine the properties of the composite graphs. It is therefore desirable to reduce the problem of computing the graph parameters (alliance numbers, for instance) of product graphs, to the problem of computing some parameters of the factor graphs.

Nowadays the study of the behavior of several graph parameters in product graphs has become an interesting topic of research [9,8]. For instance, we emphasize the Shannon capacity of a graph [15], which is a certain limiting value involving the vertex independence number of strong product powers of a graph, and Hedetniemi's coloring conjecture for the categorical product [7,8], which states that the chromatic number of any categorical product graph is equal to the minimum value between the chromatic numbers of its factors. Also, one of the oldest open problems on domination in graphs is related to Cartesian product graphs. The problem was presented first by Vizing in 1963 [19,18]. Vizing's conjecture states that the domination number of any Cartesian product graph is greater than or equal to the product of the domination numbers of its factors.

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Cartesian product graphs have been much studied in graph theory. Interest in Cartesian product graphs has been increased by the advent of massively parallel computers whose structure is that of the Cartesian product graphs [5,10]. This not only provides potential applications for the existing theory, but also suggests some new aspects of these graphs that deserve study.

Studies on defensive alliances in product graphs were initiated in [3], for the case of the torus graph $C_s \square C_t$, and studied further in [2,17,20]. Nevertheless the greater part of the results in these works are upper bounds on the alliance numbers of Cartesian product graphs. In the present work we obtain closed formulas for the global offensive alliance number of several families of Cartesian product graphs, we obtain new formulas relating the global offensive alliance number of Cartesian product graphs with the domination number and the global offensive alliance number of its factors, we present a Vizing-like conjecture for the global offensive alliance number and we prove it for several families of graphs.

We begin by stating the terminology used. Throughout this article, $G = (V, E)$ denotes a simple graph of order $|V| = n$. We denote two adjacent vertices u and v by $u \sim v$. Given a vertex $v \in V$, the set $N(v) = \{u \in V : u \sim v\}$ is the open neighborhood of v , and the set $N[v] = N(v) \cup \{v\}$ is the closed neighborhood of v . So, the degree of a vertex $v \in V$ is $d(v) = |N(v)|$.

For a nonempty set $S \subseteq V$, and a vertex $v \in V$, $N_S(v)$ denotes the set of neighbors v has in S , i.e., $N_S(v) = S \cap N(v)$. The degree of v in S will be denoted by $\delta_S(v) = |N_S(v)|$. The complement of a set S in V is denoted by \bar{S} .

A set $S \subseteq V$ is a *dominating set* in G if for every vertex $v \in \bar{S}$, $\delta_S(v) > 0$ (every vertex in \bar{S} is adjacent to at least one vertex in S). The *domination number* of G , denoted by $\gamma(G)$, is the minimum cardinality of a dominating set in G [6]. An *efficient dominating set* is a dominating set $S = \{u_1, u_2, \dots, u_{\gamma(G)}\}$ such that $N[u_i] \cap N[u_j] = \emptyset$, for every $i, j \in \{1, \dots, \gamma(G)\}$, $i \neq j$. Examples of graphs having an efficient dominating set are the path graphs P_n , the cycle graphs C_{3k} and the cube graph Q_3 .

A nonempty set $S \subseteq V$ is a *global offensive alliance* in G if

$$\delta_S(v) \geq \delta_{\bar{S}}(v) + 1, \quad \forall v \in \bar{S} \quad (1)$$

or, equivalently,

$$d(v) \geq 2\delta_{\bar{S}}(v) + 1, \quad \forall v \in \bar{S}. \quad (2)$$

Note that every global offensive alliance is a dominating set. The *global offensive alliance number* of G , denoted by $\gamma_o(G)$, is defined as the minimum cardinality of a global offensive alliance in G . A global offensive alliance of cardinality $\gamma_o(G)$ is called a $\gamma_o(G)$ -set.

We recall that given two graphs G and H with the set of vertices $V_1 = \{v_1, v_2, \dots, v_{n_1}\}$ and $V_2 = \{u_1, u_2, \dots, u_{n_2}\}$, respectively, the Cartesian product of G and H is the graph $G \square H = (V, E)$, where $V = V_1 \times V_2$ and two vertices (v_i, u_j) and (v_k, u_l) are adjacent in $G \square H$ if and only if

- $v_i = v_k$ and $u_j \sim u_l$, or
- $v_i \sim v_k$ and $u_j = u_l$.

Given two graphs $G = (V_1, E_1)$, $H = (V_2, E_2)$ and a set $X \subset V_1 \times V_2$ of vertices of $G \square H$, the projections of X over V_1 and V_2 are denoted by $P_G(X)$ and $P_H(X)$, respectively. Moreover, given a set $C \subset V_1$ of vertices of G and a vertex $v \in V_2$, a $G(C, v)$ -cell in $G \square H$ is the set $C^v = \{(u, v) \in V : u \in C\}$. A v -fiber G_v is the copy of G corresponding to the vertex v of H . For every $v \in V_2$ and $D \subset V_1 \times V_2$, let D_v be the set of vertices of D belonging to the same v -fiber.

Now we establish a Vizing-like conjecture for the global offensive alliance number.

Conjecture 1 (Vizing-Like Conjecture for the Global Offensive Alliances). *For any graphs G and H ,*

$$\gamma_o(G \square H) \geq \gamma_o(G)\gamma_o(H).$$

Below we will prove the conjecture for several families of graphs.

2. Results

Theorem 2. *For any graphs G and H ,*

$$\gamma_o(G \square H) \geq \frac{1}{2} \max\{\gamma(G)\gamma_o(H), \gamma_o(G)\gamma(H)\}.$$

Moreover, if G has an efficient dominating set, then

$$\gamma_o(G \square H) \geq \gamma(G)\gamma_o(H).$$

Proof. Let V_1 and V_2 be the vertex sets of the graphs G and H , respectively. Let $S = \{u_1, \dots, u_{\gamma(G)}\}$ be a dominating set for G . Let $\Pi = \{A_1, A_2, \dots, A_{\gamma(G)}\}$ be a vertex partition of G such that $u_i \in A_i$ and $A_i \subseteq N[u_i]$. Let $\{\Pi_1, \Pi_2, \dots, \Pi_{\gamma(G)}\}$ be a vertex partition of $G \square H$, such that $\Pi_i = A_i \times V_2$ for every $i \in \{1, \dots, \gamma(G)\}$.

Let D be a $\gamma_o(G \square H)$ -set. Now, for every $i \in \{1, \dots, \gamma(G)\}$, let $W_i = P_H(D \cap \Pi_i)$. We have

$$\gamma_o(G \square H) = |D| \geq \sum_{i=1}^{\gamma(G)} |W_i|. \quad (3)$$

If W_i is not a global offensive alliance in H , then there exists at least a vertex $v \in \overline{W_i}$ such that

$$\delta_{W_i}(v) < \delta_{\overline{W_i}}(v) + 1. \quad (4)$$

So, every vertex belonging to A_i^v has at least one neighbor in $D_v \cap \Pi_j$, for some $j \neq i$. For every $v \in V_2$, let $A_{j_1}^v, A_{j_2}^v, \dots, A_{j_{q_v}}^v$ be the $G(A_{j_i}, v)$ -cells for which v satisfies (4) and let $Y_v = S - \{u_{j_1}, u_{j_2}, \dots, u_{j_{q_v}}\}$. Since Y_v dominates $V_1 \times \{v\} - \bigcup_{i=1}^{q_v} A_{j_i}^v$ and D_v dominates $\bigcup_{i=1}^{q_v} A_{j_i}^v$, we have that $S_v = D_v \cup Y_v$ is a dominating set in the v -fiber G_v .

Now, for every $i \in \{1, \dots, \gamma(G)\}$, let $Q_i \subseteq \overline{W_i}$ be the set of vertices of H satisfying the inequality (4). Since $W_i \cup Q_i$ is a global offensive alliance in H ,

$$\gamma_o(H) \leq |W_i| + |Q_i|. \quad (5)$$

Hence, we have that

$$\begin{aligned} \gamma_o(G \square H) &\geq \sum_{i=1}^{\gamma(G)} |W_i| \\ &\geq \sum_{i=1}^{\gamma(G)} (\gamma_o(H) - |Q_i|) \\ &= \gamma(G)\gamma_o(H) - \sum_{i=1}^{\gamma(G)} |Q_i|, \end{aligned}$$

and, as a consequence, we have

$$\gamma_o(G \square H) \geq \gamma(G)\gamma_o(H) - \sum_{i=1}^{\gamma(G)} |Q_i|. \quad (6)$$

On the other hand, notice that for each $v \in V_2$, q_v is the number of $G(A_i, v)$ -cells for which v satisfies (4), as well as for each $i \in \{1, \dots, \gamma(G)\}$, $|Q_i|$ is the number of vertices of H satisfying inequality (4). Thus,

$$\sum_{v \in V_2} q_v = \sum_{i=1}^{\gamma(G)} |Q_i|. \quad (7)$$

Now, if $q_v > |D_v|$, then we have

$$\begin{aligned} |S_v| &= |D_v| + |Y_v| \\ &= |S| - q_v + |D_v| \\ &= \gamma(G) - q_v + |D_v| \\ &< \gamma(G) - q_v + q_v \\ &= \gamma(G), \end{aligned}$$

which is a contradiction. So, we have $q_v \leq |D_v|$ and we obtain

$$\sum_{v \in V_2} q_v \leq \sum_{v \in V_2} |D_v| = \gamma_o(G \square H). \quad (8)$$

Thus, by (6)–(8) we deduce

$$\gamma_o(G \square H) \geq \gamma(G)\gamma_o(H) - \gamma_o(G \square H).$$

Analogously, we obtain that $\gamma_o(G \square H) \geq \gamma_o(G)\gamma(H) - \gamma_o(G \square H)$. Therefore, the first result follows.

Now, if $S = \{u_1, \dots, u_{\gamma(G)}\}$ is an efficient dominating set for G , then for every $i \in \{1, \dots, \gamma(G)\}$, W_i is a global offensive alliance in H . That is, if we suppose that W_i is not a global offensive alliance in H , then there exists at least one vertex $v \in \overline{W_i}$ which satisfies (4). Thus, every vertex belonging to A_i^v has at least one neighbor in $D_v \cap \Pi_j$, for some $j \neq i$, which is a contradiction because (u_i, v) has no neighbors outside of Π_i . As a consequence, $|Q_i| = 0$. So, (3) and (5) directly lead to $\gamma_o(G \square H) \geq \gamma(G)\gamma_o(H)$. \square

Notice that for the case of star graphs, $S_{1,n}$, the central vertex forms an efficient dominating set of minimum cardinality, and it is also a global offensive alliance, then the above theorem leads to the following Vizing-like result for the global offensive alliance number.

Corollary 3. Let $S_{1,n}$ be a star graph. For any graph H ,

$$\gamma_o(S_{1,n} \square H) \geq \gamma_o(S_{1,n})\gamma_o(H).$$

As the following remark shows there is no other family of connected graphs containing an efficient dominating set of minimum cardinality which is also a global offensive alliance.

Remark 4. A connected graph G contains an efficient dominating set of minimum cardinality which is also a global offensive alliance if and only if G is a star graph.

Proof. If G is a star graph, then it is clear that the central vertex is an efficient dominating set of minimum cardinality, and also a global offensive alliance.

On the contrary, suppose G is not a star graph and let $S = \{u_1, u_2, \dots, u_{\gamma(G)}\}$ be an efficient dominating set of minimum cardinality which is also a global offensive alliance in G . So, for every $u_i, u_j \in S$, $i \neq j$, we have that $N[u_i] \cap N[u_j] = \emptyset$. As a consequence, for every $v \in \bar{S}$ we have $\delta_S(v) = 1$ and by inequality (1) we have, $\delta_S(v) \geq \delta_{\bar{S}}(v) + 1$. Hence, $\delta_{\bar{S}}(v) = 0$ and so the degree of v in G is one, i.e., every vertex outside of S is an end-vertex. Now, if $\gamma(G) \geq 2$, then since for every $u_i, u_j \in S$, $i \neq j$, we have that $N[u_i] \cap N[u_j] = \emptyset$ we obtain that $u_i \sim u_j$. So, G is not connected, which is a contradiction. Therefore $\gamma(G) = 1$ and, as a consequence, G is a star graph. \square

Theorem 5. Let P_n be a path graph of order n . For every graph G of minimum degree $\delta \geq 1$,

$$\gamma_o(G \square P_n) \geq \left\lceil \frac{(n-1)\gamma_o(G)}{2} \right\rceil + \left\lceil \frac{\delta}{2} \right\rceil.$$

Proof. Let S be a $\gamma_o(G \square P_n)$ -set. Let $\{v_1, v_2, \dots, v_n\}$ be the set of vertices of P_n . Let V_i be the vertex set of the v_i -fiber G_{v_i} and let $S_i = P_G(S \cap V_i)$.

For every $(x, v_1) \notin S$ we have $\delta_S(x, v_1) \geq \delta_{\bar{S}}(x, v_1) + 1$. Adding $\delta_S(x, v_1)$ to both sides of this inequality we have $2\delta_S(x, v_1) \geq d(x, v_1) + 1$. Hence,

$$2(|S_1| + 1) \geq 2(\delta_{S_1}(x) + 1) \geq 2\delta_S(x, v_1) \geq d(x, v_1) + 1 = d(x) + 2 \geq \delta + 2.$$

As a consequence, $|S_1| \geq \left\lceil \frac{\delta}{2} \right\rceil$. Analogously we show that $|S_n| \geq \left\lceil \frac{\delta}{2} \right\rceil$.

We suppose there exists $i \in \{1, 2, \dots, n-1\}$ such that S_i is not a global offensive alliance in G . Let $S'_i \subset V_i - S_i$ such that $\delta_{S_i}(x) < \delta_{\bar{S}_i}(x) + 1$, for every $x \in S'_i$. Now let $x \in S'_i$ and suppose $(x, v_{i+1}) \notin S$. If $i = 1$, then $\delta_S(x, v_1) = \delta_{S_1}(x) < \delta_{\bar{S}_1}(x) + 1 = \delta_{\bar{S}}(x, v_1)$, a contradiction. If $1 < i < n$, then $\delta_S(x, v_i) \leq \delta_{S_i}(x) + 1 < \delta_{\bar{S}_i}(x) + 2 \leq \delta_{\bar{S}}(x, v_i) + 1$, also a contradiction. Hence, if S_i is not a global offensive alliance in G , then for every $x \in S'_i$ we have $(x, v_{i+1}) \in S$. As a consequence, $S_i \cup S'_i$ is a global offensive alliance in G and for every $i \in \{1, 2, \dots, n-1\}$, $|S_i \cup S_{i+1}| \geq |S_i \cup S'_i| \geq \gamma_o(G)$. So, $\sum_{i=1}^{n-1} |S_i \cup S_{i+1}| \geq (n-1)\gamma_o(G)$ and we have

$$2|S| = 2 \sum_{i=1}^n |S_i| = \sum_{i=1}^{n-1} |S_i \cup S_{i+1}| + |S_1| + |S_n| \geq (n-1)\gamma_o(G) + |S_1| + |S_n| \geq (n-1)\gamma_o(G) + 2 \left\lceil \frac{\delta}{2} \right\rceil.$$

Therefore, $|S| \geq \frac{(n-1)\gamma_o(G)}{2} + \left\lceil \frac{\delta}{2} \right\rceil$. \square

We note that since $\gamma_o(P_n) = \left\lfloor \frac{n}{2} \right\rfloor$, the above theorem leads to the following corollary.

Corollary 6. Let P_{2k+1} be a path graph of odd order. For any graph G ,

$$\gamma_o(G \square P_{2k+1}) > \gamma_o(G)\gamma_o(P_{2k+1}).$$

Theorem 7. Let C_n be a cycle graph of order n . For every graph G ,

$$\gamma_o(G \square C_n) \geq \left\lceil \frac{n\gamma_o(G)}{2} \right\rceil.$$

Proof. Let S be a $\gamma_o(G \square C_n)$ -set. Let $\{v_0, v_2, \dots, v_{n-1}\}$ be the set of vertices of C_n . Let V_i be the vertex set of the v_i -fiber G_{v_i} and let $S_i = P_G(S \cap V_i)$. Proceeding as in the proof of Theorem 5 we show that for every $i \in \{0, 1, \dots, n-1\}$, $|S_i \cup S_{i+1}| \geq \gamma_o(G)$,

where the subscripts are taken modulo n . Hence,

$$2|S| = \sum_{i=0}^{n-1} |S_i \cup S_{i+1}| \geq n\gamma_o(G).$$

Therefore, the result follows. \square

In order to deduce an upper bound on $\gamma_o(G \square H)$ we are going to introduce two known results. It was shown in [16] that for every connected graph G of order $n \geq 2$ and independence number $\alpha(G)$, it follows that

$$\gamma_o(G) + \alpha(G) \leq n. \quad (9)$$

The *eccentricity* of a vertex v of a connected graph G is the maximum distance between v and any other vertex u of G . The *radius* of G is the minimum eccentricity of any vertex in G . It was shown in [1] that for every connected graphs G and H of radius $r(G)$ and $r(H)$, it follows that

$$\alpha(G \square H) \geq 2r(G)r(H). \quad (10)$$

As a direct consequence of (9) and (10) we have the following bound.

Proposition 8. For any connected graphs G and H of order n_1 and n_2 and radius $r(G)$ and $r(H)$, respectively,

$$\gamma_o(G \square H) \leq n_1 n_2 - 2r(G)r(H).$$

The above bound is tight. It is achieved, for instance, for the torus graphs $C_{2k} \square C_{2k'}$ (see Proposition 17), for the grid graphs $P_{2k} \square P_{2k'}$ (see Proposition 22) and for the cylinder graphs $P_{2k} \square C_{2k'}$ (see Proposition 24).

We derive another upper bound on $\gamma_o(G \square H)$ from (9) and the following bound on $\alpha(G \square H)$ obtained in [9] for every bipartite graph G of order n :

$$\alpha(G \square H) \geq \frac{n}{2} \alpha_2(H), \quad (11)$$

where $\alpha_2(H)$ is the bipartite number of H , i.e., the order of the largest induced bipartite subgraph of H .

Proposition 9. For any connected bipartite graph G of order n_1 and any connected graph H of order n_2 and bipartite number $\alpha_2(H)$,

$$\gamma_o(G \square H) \leq n_1 \left(n_2 - \frac{\alpha_2(H)}{2} \right).$$

As a direct consequence of Proposition 9 we obtain the following result.

Corollary 10. Let G and H be two connected bipartite graphs of order n_1 and n_2 , respectively. Then $\gamma_o(G \square H) \leq \frac{n_1 n_2}{2}$.

The above bound is tight. It is achieved, for instance, for the torus graphs $C_{2k} \square C_{2k'}$ (see Proposition 17), for the grid graphs $P_{2k} \square P_{2k'}$ (see Proposition 22) and for the cylinder graphs $P_{2k} \square C_{2k'}$ (see Proposition 24).

We recall that a graph $H = (V, E)$ is partitionable into two global offensive alliances if there exists a partition $\{Y_1, Y_2\}$ of V such that both Y_1 and Y_2 are global offensive alliances in H [17].

Theorem 11. Let H be a graph of order n . If H is partitionable into two global offensive alliances, then

$$\gamma_o(K_r \square H) \leq \left\lfloor \frac{rn}{2} \right\rfloor.$$

Proof. Let $U = \{u_1, u_2, \dots, u_r\}$ and $V = \{v_1, v_2, \dots, v_n\}$ be the sets of vertices of K_r and H , respectively. Let $\{Y_1, Y_2\}$ be a partition of V such that both Y_1 and Y_2 are global offensive alliances in H , where $|Y_1| \leq |Y_2|$. Let $X_1 = \{u_1, u_2, \dots, u_{\lceil \frac{r}{2} \rceil}\}$ and let $X_2 = \{u_{\lceil \frac{r}{2} \rceil + 1}, u_{\lceil \frac{r}{2} \rceil + 2}, \dots, u_r\}$. Let us show that $S = (X_1 \times Y_1) \cup (X_2 \times Y_2)$ is a global offensive alliance in $K_r \square H$. If $(u, v) \in X_1 \times Y_2$, then

$$\begin{aligned} \delta_S(u, v) &= \delta_{X_2}(u) + \delta_{Y_1}(v) \\ &= \left\lfloor \frac{r}{2} \right\rfloor + \delta_{Y_1}(v) \\ &\geq \left\lfloor \frac{r}{2} \right\rfloor - 1 + \delta_{Y_1}(v) \\ &\geq \left\lfloor \frac{r}{2} \right\rfloor - 1 + \delta_{Y_2}(v) + 1 \\ &= \delta_{X_1}(u) + \delta_{Y_2}(v) + 1 \\ &= \delta_{\bar{S}}(u, v) + 1. \end{aligned}$$

Now, if $(u, v) \in X_2 \times Y_1$, then

$$\begin{aligned}\delta_S(u, v) &= \delta_{X_1}(u) + \delta_{Y_2}(v) \\ &= \left\lceil \frac{r}{2} \right\rceil + \delta_{Y_2}(v) \\ &\geq \left\lceil \frac{r}{2} \right\rceil + \delta_{Y_1}(v) + 1 \\ &\geq \left\lfloor \frac{r}{2} \right\rfloor - 1 + \delta_{Y_1}(v) + 1 \\ &= \delta_{X_2}(u) + \delta_{Y_1}(v) + 1 \\ &= \delta_{\bar{S}}(u, v) + 1.\end{aligned}$$

Hence, $S = (X_1 \times Y_1) \cup (X_2 \times Y_2)$ is a global offensive alliance in $K_r \square H$. If r is even, then $|S| = \frac{m}{2}$ and, if r is odd, then $|S| = \frac{r+1}{2}|Y_1| + \frac{r-1}{2}|Y_2| = \frac{r}{2}n + \frac{|Y_1|-|Y_2|}{2} \leq \frac{r}{2}n$. The proof is complete. \square

Proposition 12. The global offensive alliance number of the bamboo graph $K_r \square P_t$ is

$$\gamma_o(K_r \square P_t) = \left\lfloor \frac{rt}{2} \right\rfloor.$$

Proof. Since P_t is partitionable into two global offensive alliances, by Theorem 11 we obtain the upper bound $\gamma_o(K_r \square P_t) \leq \left\lfloor \frac{rt}{2} \right\rfloor$.

On the other hand, let S be a $\gamma_o(K_r \square P_t)$ -set. Let $\{u_1, u_2, \dots, u_r\}$ and $\{v_1, v_2, \dots, v_t\}$ be the sets of vertices of K_r and P_t , respectively. In P_t adjacent vertices have consecutive subscripts. Now, let V_j be the vertex set of the v_j -fiber and let $S_j = P_{K_r}(S \cap V_j)$. We first note that since S is a global offensive alliance in $K_r \square P_t$, for every $(u_i, v_1) \in V_1$ it follows $1 + |S_1| \geq \delta_S(u_i, v_1) \geq \frac{r+1}{2}$, so $|S_1| \geq \left\lceil \frac{r-1}{2} \right\rceil = \left\lfloor \frac{r}{2} \right\rfloor$. Analogously, $|S_t| \geq \left\lfloor \frac{r}{2} \right\rfloor$. Now, suppose there exists $j \in \{1, \dots, t-1\}$ such that $|S_j \cup S_{j+1}| < r$. In such a case, $|S_j| \leq \frac{r-1}{2}$ or $|S_{j+1}| \leq \frac{r-1}{2}$, and there exist $(u_i, v_j), (u_i, v_{j+1}) \notin S$. We take, without loss of generality, $|S_j| \leq \frac{r-1}{2}$. Thus,

$$1 + \frac{r-1}{2} \geq \delta_S(u_i, v_j) \geq \delta_{\bar{S}}(u_i, v_j) + 1 \geq \frac{r-1}{2} + 2,$$

a contradiction. Hence, for every $j \in \{1, 2, \dots, t-1\}$, $|S_j \cup S_{j+1}| \geq r$. As a consequence

$$2|S| = \sum_{j=1}^t |S_j| \geq r(t-1) + |S_1| + |S_t| \geq r(t-1) + 2 \left\lfloor \frac{r}{2} \right\rfloor.$$

Thus,

$$\gamma_o(K_r \square P_t) \geq \left\lceil \frac{r(t-1)}{2} \right\rceil + \left\lfloor \frac{r}{2} \right\rfloor = \left\lfloor \frac{rt}{2} \right\rfloor.$$

Therefore, the proof is complete. \square

Corollary 13. For any complete graph K_r and any path graph P_t ,

$$\gamma_o(K_r \square P_t) \geq \gamma_o(K_r) \gamma_o(P_t).$$

We recall that a set S of vertices of a graph H is a global strong offensive alliance if for every $x \in \bar{S}$, $\delta_S(x) \geq \delta_{\bar{S}}(x)$. Note that every global offensive alliance is a global strong offensive alliance. It was shown in [17] that every graph without isolated vertices is partitionable into global strong offensive alliances.

Theorem 14. Let G be a graph partitionable into two global offensive alliances X_1 and X_2 and let H be a graph partitionable into two global strong offensive alliances Y_1 and Y_2 . Then

$$\gamma_o(G \square H) \leq |X_1| |Y_1| + |X_2| |Y_2|.$$

Proof. Let $U = \{u_1, u_2, \dots, u_{n_1}\}$ and $V = \{v_1, v_2, \dots, v_{n_2}\}$ be the sets of vertices of G and H , respectively, where $X_1 = \{u_1, u_2, \dots, u_{|X_1|}\}$, $X_2 = \{u_{|X_1|+1}, u_{|X_1|+2}, \dots, u_{n_1}\}$, $Y_1 = \{v_1, v_2, \dots, v_{|Y_1|}\}$ and $Y_2 = \{v_{|Y_1|+1}, v_{|Y_1|+2}, \dots, v_{n_2}\}$. Let us show that $S = (X_1 \times Y_1) \cup (X_2 \times Y_2)$ is a global offensive alliance in $G \square H$. If $(u, v) \in X_1 \times Y_2$, then

$$\begin{aligned}\delta_S(u, v) &= \delta_{X_2}(u) + \delta_{Y_1}(v) \\ &\geq \delta_{X_1}(u) + \delta_{Y_2}(v) + 1 \\ &\geq \delta_{\bar{S}}(u, v) + 1.\end{aligned}$$

Now, if $(u, v) \in X_2 \times Y_1$, then

$$\begin{aligned}\delta_S(u, v) &= \delta_{X_1}(u) + \delta_{Y_2}(v) \\ &\geq \delta_{X_2}(u) + \delta_{Y_1}(v) + 1 \\ &\geq \delta_{\bar{S}}(u, v) + 1.\end{aligned}$$

Hence, $S = (X_1 \times Y_1) \cup (X_2 \times Y_2)$ is a global offensive alliance in $G \square H$. The proof is complete. \square

The proof of the following result is completely analogous to the proof of [Theorem 14](#).

Theorem 15. Let G be a graph partitionable into a global offensive alliance X_1 and a global strong offensive alliance X_2 . Let H be a graph partitionable into a global offensive alliance Y_1 and a global strong offensive alliance Y_2 . Then

$$\gamma_o(G \square H) \leq |X_1| |Y_1| + |X_2| |Y_2|.$$

A bipartite graph $G = (X_1 \cup X_2, E)$, where the sets of the bipartition have cardinality $|X_1| = x_1$ and $|X_2| = x_2$ is called a (x_1, x_2) -bipartite graph.

Corollary 16. Let G be a (p_1, p_2) -bipartite graph and let H be a (t_1, t_2) -bipartite graph. Then

$$\gamma_o(G \square H) \leq p_1 t_1 + p_2 t_2.$$

We recall that the hypercube graphs are defined as $Q_k = Q_{k-1} \square K_2$, $k \geq 2$, where $Q_1 = K_2$. Note that Q_{k-1} is a $(2^{k-2}, 2^{k-2})$ -bipartite graph and K_2 is a $(1, 1)$ -bipartite graph. Moreover, the Laplacian spectral radius of Q_k is $\lambda = 2k$. Hence, from the above corollary and [\(12\)](#) we have

$$\left\lceil \left\lceil \frac{k+1}{2} \right\rceil \frac{2^{k-1}}{k} \right\rceil \leq \gamma_o(Q_k) \leq 2^{k-1}.$$

Proposition 17. The global offensive alliance number of the torus graph $C_r \square C_t$ is

$$\gamma_o(C_r \square C_t) = \left\lceil \frac{rt}{2} \right\rceil.$$

Proof. Since every cycle C_n can be partitioned into a global strong offensive alliance of cardinality $\left\lfloor \frac{n}{2} \right\rfloor$ and a global offensive alliance of cardinality $\left\lceil \frac{n}{2} \right\rceil$, by [Theorem 15](#) we have $\gamma_o(C_r \square C_t) \leq \left\lceil \frac{rt}{2} \right\rceil$.

On the other hand, let S be a $\gamma_o(C_r \square C_t)$ -set. Let $\{u_0, u_1, \dots, u_{r-1}\}$ and $\{v_0, v_1, \dots, v_{t-1}\}$ be the sets of vertices of C_r and C_t , respectively. Here adjacent vertices have consecutive subscripts, where the subscripts are taken modulo r and t , respectively. As above, let V_j be the vertex set of the v_j -fiber and let $S_j = P_{C_r}(S \cap V_j)$. Let (u_i, v_j) be a vertex not belonging to S . Since $C_r \square C_t$ is a 4-regular graph and S is a global offensive alliance, if $(u_{i+1}, v_j) \notin S$, then $(u_i, v_{j+1}), (u_{i+1}, v_{j+1}) \in S$, and if $(u_i, v_{j+1}) \notin S$, then $(u_{i+1}, v_j), (u_{i+1}, v_{j+1}) \in S$. Thus, for every $j \in \{0, 1, \dots, t-1\}$, $|S_j \cup S_{j+1}| \geq r$. Hence,

$$2|S| = \sum_{j=0}^{t-1} |S_j \cup S_{j+1}| \geq rt.$$

Therefore, we have that $\gamma_o(C_r \square C_t) \geq \left\lceil \frac{rt}{2} \right\rceil$ and the proof is complete. \square

Corollary 18. For any torus graph $C_r \square C_t$, $\gamma_o(C_r \square C_t) \geq \gamma_o(C_r) \gamma_o(C_t)$.

Proposition 19. The global offensive alliance number of the graph $K_r \square C_t$ is

$$\gamma_o(K_r \square C_t) = \left\lceil \frac{rt}{2} \right\rceil.$$

Proof. Let S be a $\gamma_o(K_r \square C_t)$ -set. Let $\{u_1, u_2, \dots, u_r\}$ and $\{v_0, v_1, \dots, v_{t-1}\}$ be the sets of vertices of K_r and C_t , respectively. In C_t the subscripts are taken modulo t and adjacent vertices have consecutive subscripts. As above, let V_j be the vertex set of the v_j -fiber and let $S_j = P_{K_r}(S \cap V_j)$. Let (u_i, v_j) be a vertex not belonging to S .

Now, suppose there exists $j \in \{0, 1, \dots, t\}$ such that $|S_j \cup S_{j+1}| < r$. In such a case, $|S_j| \leq \frac{r-1}{2}$ or $|S_{j+1}| \leq \frac{r-1}{2}$, and there exist $(u_i, v_j), (u_i, v_{j+1}) \notin S$. We take, without loss of generality, $|S_j| \leq \frac{r-1}{2}$. Thus,

$$1 + \frac{r-1}{2} \geq \delta_S(u_i, v_j) \geq \delta_{\bar{S}}(u_i, v_j) + 1 \geq \frac{r-1}{2} + 2,$$

a contradiction. Hence, for every $j \in \{1, 2, \dots, t-1\}$, $|S_j \cup S_{j+1}| \geq r$ (the subscripts are taken modulo t). As a consequence $2|S| = \sum_{j=0}^{t-1} |S_j \cup S_{j+1}| \geq rt$. Therefore, $\gamma_o(K_r \square C_t) \geq \left\lceil \frac{rt}{2} \right\rceil$.

Since every cycle graph C_n (every complete graph K_n) can be partitioned into a global strong offensive alliance of cardinality $\lfloor \frac{n}{2} \rfloor$ and a global offensive alliance of cardinality $\lceil \frac{n}{2} \rceil$, by Theorem 15 we have $\gamma_o(K_r \square C_t) \leq \lceil \frac{rt}{2} \rceil$. Therefore, the proof is complete. \square

Corollary 20. For any complete graph K_r and any cycle graph C_t ,

$$\gamma_o(K_r \square C_t) \geq \gamma_o(K_r) \gamma_o(C_t).$$

A square in a Cartesian product of two graphs G and H is a set of vertices of $G \square H$ formed by four different vertices $(u_i, v_k), (u_i, v_l), (u_j, v_k), (u_j, v_l)$ such that $u_i \sim u_j$ in G and $v_k \sim v_l$ in H .

Lemma 21. Let G and H be two graphs such that they are cycles or paths and let S be a $\gamma_o(G \square H)$ -set. For any square A of $G \square H$ it follows that $|S \cap A| \geq 2$.

Proof. The result follows directly from the fact that if at least three of the vertices of the square $A = \{(u_i, v_k), (u_i, v_l), (u_j, v_k), (u_j, v_l)\}$ do not belong to S , then (at least) for one of these three vertices, say (u_i, v_k) , it is satisfied that $\delta_{\bar{S}}(u_i, v_k) \geq 2$, which is a contradiction because $G \square H$ has maximum degree four and by (2) we know that for every $(u, v) \in \bar{S}$, it follows $\delta_{\bar{S}}(u, v) \leq 1$. \square

Proposition 22. Let $P_r \square P_t$ be a grid graph.

- (i) If r and t are even, then $\gamma_o(P_r \square P_t) = \frac{rt}{2}$.
- (ii) If r is even and t is odd, then $\gamma_o(P_r \square P_t) = \frac{r(t-1)}{2} + \lceil \frac{r}{3} \rceil$.
- (iii) If r and t are odd, then $\frac{(r-1)(t-1)}{2} + \lceil \frac{r}{3} \rceil + \lceil \frac{t}{3} \rceil \leq \gamma_o(P_r \square P_t) \leq \frac{r(t-1)}{2} + \lceil \frac{r}{3} \rceil$.

Proof. Let $V_1 = \{u_1, u_2, \dots, u_r\}$ and $V_2 = \{v_1, v_2, \dots, v_t\}$ be the sets of vertices of P_r and P_t , respectively. Here adjacent vertices have consecutive subscripts. Let S be a $\gamma_o(P_r \square P_t)$ -set.

Suppose r and t are even. Since there exists a vertex partition of $P_r \square P_t$ into $\frac{rt}{4}$ disjoint squares, by Lemma 21 we have that $\gamma_o(P_r \square P_t) = |S| \leq \frac{rt}{2}$. Moreover, by Proposition 8 we have $\gamma_o(P_r \square P_t) = |S| \leq \frac{rt}{2}$. Therefore, (i) follows.

Now we suppose r is even and t is odd. Since there exists a vertex partition of $P_r \square P_{t-1}$ into $\frac{r(t-1)}{4}$ disjoint squares, by Lemma 21 we have that $\gamma_o(P_r \square P_t) \geq \frac{r(t-1)}{2} + |S_t|$.

As above, let V_t be the set of vertices of the P_r -fiber corresponding to the vertex v_t of P_t and let $S_t = S \cap V_t$. Notice that if a vertex of V_t , of degree two in $P_r \square P_t$, does not belong to S , then its two neighbors belong to S . Also, if three vertices of V_t induce a path in $P_r \square P_t$, then at least one of them belongs to S . Thus, we have that $|S_t| \geq \lceil \frac{r}{3} \rceil$ and, as a consequence, $\gamma_o(P_r \square P_t) \geq \frac{r(t-1)}{2} + |S_t| \geq \frac{r(t-1)}{2} + \lceil \frac{r}{3} \rceil$.

On the other hand, let W be the subset of vertices of $P_r \square P_t$ taken in the following way. If $r \equiv 0 \pmod{3}$, then W is composed of pairs (u_i, v_j) with $i \in \{2, 5, 8, \dots, r-1\}$ and $j \in \{1, 3, 5, \dots, t-2, t\}$ as well as pairs (u_i, v_j) with $i \in \{1, 3, 4, 6, 7, \dots, r-3, r-2, r\}$ and $j \in \{2, 4, 6, \dots, t-3, t-1\}$.

Notice that in this case

$$|W| = \frac{t-1}{2} \frac{r}{3} + \frac{t-1}{2} \left(r - \frac{r}{3}\right) + \frac{r}{3} = \frac{r(t-1)}{2} + \frac{r}{3} = \frac{r(t-1)}{2} + \lceil \frac{r}{3} \rceil.$$

If $r \equiv 1 \pmod{3}$, then W is composed of pairs (u_i, v_j) with $i \in \{1, 4, 7, \dots, r-6, r-3, r\}$ and $j \in \{1, 3, 5, \dots, t-2, t\}$ as well as pairs (u_i, v_j) with $i \in \{2, 3, 5, 6, \dots, r-2, r-1\}$ and $j \in \{2, 4, 6, \dots, t-3, t-1\}$.

If $r \equiv 2 \pmod{3}$, then W is composed of pairs (u_i, v_j) with $i \in \{1, 4, 7, \dots, r-7, r-4, r-1\}$ and $j \in \{1, 3, 5, \dots, t-2, t\}$ as well as pairs (u_i, v_j) with $i \in \{2, 3, 5, 6, \dots, r-3, r-2, r\}$ and $j \in \{2, 4, 6, \dots, t-3, t-1\}$.

Thus, in the above two cases ($r \equiv 1 \pmod{3}$ or $r \equiv 2 \pmod{3}$) we have

$$|W| = \frac{t-1}{2} \lceil \frac{r}{3} \rceil + \frac{t-1}{2} \left(r - \lceil \frac{r}{3} \rceil\right) + \lceil \frac{r}{3} \rceil = \frac{r(t-1)}{2} + \lceil \frac{r}{3} \rceil.$$

So, in all the above cases we have that $|W| = \frac{r(t-1)}{2} + \lceil \frac{r}{3} \rceil$. Moreover, for every vertex $(u, v) \notin W$ we have that $\delta_W(u, v) = 2 \geq 1 = \delta_{\bar{W}}(u, v) + 1$, if (u, v) has degree two, $\delta_W(u, v) = 3 \geq 1 = \delta_{\bar{W}}(u, v) + 1$, if (u, v) has degree three, and $\delta_W(u, v) = 3 \geq 2 = \delta_{\bar{W}}(u, v) + 1$, if (u, v) has degree four. Thus, W is a global offensive alliance and so, $\gamma_o(P_r \square P_t) \leq \frac{r(t-1)}{2} + \lceil \frac{r}{3} \rceil$. Therefore, (ii) follows.

Finally, we suppose r and t are odd. Since there exists a vertex partition of $P_{r-1} \square P_{t-1}$ into $\frac{(r-1)(t-1)}{4}$ disjoint squares, using a similar argument to the above case we have that $\gamma_o(P_r \square P_t) \geq \frac{(r-1)(t-1)}{2} + \lceil \frac{t}{3} \rceil + \lceil \frac{r}{3} \rceil$. On the other hand, let Q be the subset of vertices of $P_r \square P_t$ taken in the following way.

If $r \equiv 0 \pmod{3}$, then Q is composed of pairs (u_i, v_j) with $i \in \{2, 5, 8, \dots, r-1\}$ and $j \in \{1, 3, 5, \dots, t-2, t\}$ as well as pairs (u_i, v_j) with $i \in \{1, 3, 4, 6, 7, \dots, r-3, r-2, r\}$ and $j \in \{2, 4, 6, \dots, t-3, t-1\}$.

If $r \equiv 1 \pmod{3}$, then Q is composed of pairs (u_i, v_j) with $i \in \{1, 4, 7, \dots, r-6, r-3, r\}$ and $j \in \{1, 3, 5, \dots, t-2, t\}$ as well as pairs (u_i, v_j) with $i \in \{2, 3, 5, 6, \dots, r-2, r-1\}$ and $j \in \{2, 4, 6, \dots, t-3, t-1\}$.

If $r \equiv 2 \pmod{3}$, then Q is composed of pairs (u_i, v_j) with $i \in \{1, 4, 7, \dots, r-7, r-4, r-1\}$ and $j \in \{1, 3, 5, \dots, t-2, t\}$ as well as pairs (u_i, v_j) with $i \in \{2, 3, 5, 6, \dots, r-3, r-2, r\}$ and $j \in \{2, 4, 6, \dots, t-3, t-1\}$.

Thus, in the above cases we have

$$|Q| = \left\lceil \frac{r}{3} \right\rceil \left\lceil \frac{t}{2} \right\rceil + \left(r - \left\lceil \frac{r}{3} \right\rceil \right) \left\lfloor \frac{t}{2} \right\rfloor = \frac{r(t-1)}{2} + \left\lceil \frac{r}{3} \right\rceil.$$

Moreover, for every vertex $(u, v) \notin Q$ we have that $\delta_Q(u, v) = 2 \geq 1 = \delta_{\overline{Q}}(u, v) + 1$, if (u, v) has degree two, $\delta_Q(u, v) = 3 \geq 1 = \delta_{\overline{Q}}(u, v) + 1$, if (u, v) has degree three, and $\delta_Q(u, v) = 3 \geq 2 = \delta_{\overline{Q}}(u, v) + 1$, if (u, v) has degree four. Thus, Q is a global offensive alliance in $P_r \square P_t$ and, as a consequence, $\gamma_o(P_r \square P_t) \leq \frac{r(t-1)}{2} + \left\lceil \frac{r}{3} \right\rceil$. Therefore, (iii) follows. \square

Corollary 23. For any path graph P_r and any path graph P_t ,

$$\gamma_o(P_r \square P_t) \geq \gamma_o(P_r) \gamma_o(P_t).$$

Proposition 24. The global offensive alliance number of the cylinder graph $P_r \square C_t$ is

$$\gamma_o(P_r \square C_t) = \begin{cases} \frac{rt}{2}, & \text{if } r \text{ is even,} \\ \frac{(r-1)t}{2} + \left\lceil \frac{t}{3} \right\rceil, & \text{if } r \text{ is odd.} \end{cases}$$

Proof. Let S be a $\gamma_o(P_r \square C_t)$ -set. Let $\{u_1, u_2, \dots, u_r\}$ and $\{v_0, v_1, \dots, v_{t-1}\}$ be the sets of vertices of P_r and C_t , respectively (here adjacent vertices have consecutive subscripts. In the case of C_t , the subscripts are taken modulo t). We differentiate the following cases.

Case 1: r even. As above, let V_j be the vertex set of the v_j -fiber and let $S_j = P_r(S \cap V_j)$. Let (u_i, v_j) be a vertex not belonging to S . Since every vertex of $P_r \square C_t$ has degree three or four and S is a global offensive alliance, if $(u_{i+1}, v_j) \notin S$, then $(u_i, v_{j+1}), (u_{i+1}, v_{j+1}) \in S$, and if $(u_i, v_{j+1}) \notin S$, then $(u_{i+1}, v_j), (u_{i+1}, v_{j+1}) \in S$. Thus, for every $j \in \{0, 1, \dots, t-1\}$, $|S_j \cup S_{j+1}| \geq r$. Hence

$$2|S| = \sum_{j=0}^{t-1} |S_j \cup S_{j+1}| \geq rt.$$

Therefore, we have that $\gamma_o(P_r \square C_t) \geq \frac{rt}{2}$. Since every cycle C_n (every path P_n) can be partitioned into a global strong offensive alliance of cardinality $\left\lfloor \frac{n}{2} \right\rfloor$ and a global offensive alliance of cardinality $\left\lceil \frac{n}{2} \right\rceil$, by Theorem 15 we have $\gamma_o(P_r \square C_t) \leq \frac{rt}{2}$.

Case 2: r odd. The number of squares of $P_r \square C_t$ is $(r-1)t$. By Lemma 21 we know that each square of $P_r \square C_t$ contains at least two vertices belonging to S , moreover, each vertex of S belongs to four different squares, except the vertices of degree three which only belong to two different squares. So, we have $2(r-1)t \leq 4(|S| - |S'|) + 2|S'|$, where $S' = \{(u, v) \in S : d(u, v) = 3\}$. Note also that if three vertices of degree three induce a path in $P_r \square C_t$, then at least one of them belongs to S . Thus, $|S'| \geq 2 \left\lceil \frac{t}{3} \right\rceil$. Hence, $|S| \geq \frac{(r-1)t}{2} + \left\lceil \frac{t}{3} \right\rceil$.

Now, let Y be the subset of vertices of $P_r \square C_t$ which is formed by pairs (u_i, v_j) with $i \in \{1, 3, 5, \dots, r\}$ and $j \in \{k : 0 \leq k \leq t-1, k \equiv 0 \pmod{3}\}$ as well as pairs (u_i, v_j) with $i \in \{2, 4, \dots, r-1\}$ and $j \in \{k : 0 \leq k \leq t-1, k \not\equiv 0 \pmod{3}\}$.

Then, clearly $|Y| = \left\lceil \frac{r}{2} \right\rceil \left\lceil \frac{t}{3} \right\rceil + \left\lfloor \frac{r}{2} \right\rfloor \left(t - \left\lceil \frac{t}{3} \right\rceil \right) = \frac{(r-1)t}{2} + \left\lceil \frac{t}{3} \right\rceil$. Now, since for every $(u, v) \notin Y$ we have that $\delta_Y(u, v) = 0$ or $\delta_Y(u, v) = 1$, we conclude that Y is a global offensive alliance in $P_r \square C_t$ and, as a consequence, $\gamma_o(P_r \square C_t) \leq |Y| = \frac{(r-1)t}{2} + \left\lceil \frac{t}{3} \right\rceil$. Therefore, the proof is complete. \square

Corollary 25. For any path graph P_r and any cycle graph C_t ,

$$\gamma_o(P_r \square C_t) \geq \gamma_o(P_r) \gamma_o(C_t).$$

It was shown in [16] that the global offensive alliance number of a connected graph G of order n is bounded by

$$\gamma_o(G) \geq \left\lceil \frac{n}{\lambda} \left\lceil \frac{\delta + 1}{2} \right\rceil \right\rceil, \quad (12)$$

where λ is the Laplacian spectral radius¹ of G and δ its minimum degree. This bound will be useful to prove the following result.

¹ The Laplacian spectral radius of a graph G is the largest Laplacian eigenvalue of G . More information about Laplacian eigenvalues of a graph can be found in [12].

Proposition 26. Let r and t be two positive integers. If r, t have the same parity, then

$$\gamma_o(K_r \square K_t) = \left\lceil \frac{rt}{2} \right\rceil.$$

If r and t have different parity, then

$$\left\lceil \frac{rt(r+t-1)}{2(r+t)} \right\rceil \leq \gamma_o(K_r \square K_t) \leq \left\lceil \frac{rt}{2} \right\rceil.$$

Proof. Since every complete graph K_n can be partitioned into a global strong offensive alliance of cardinality $\lfloor \frac{n}{2} \rfloor$ and a global offensive alliance of cardinality $\lceil \frac{n}{2} \rceil$, by Theorem 15 we have $\gamma_o(K_r \square K_t) \leq \lceil \frac{rt}{2} \rceil$.

On the other hand, in order to apply (12) to $K_r \square K_t$, we recall that in this case we have order rt , degree $r+t-2$ and Laplacian spectral radius $\lambda = r+t$. So, if r and t have the same parity, then (12) leads to $\gamma_o(K_r \square K_t) \geq \lceil \frac{rt}{2} \rceil$, and if r and t have different parity, then (12) leads to $\gamma_o(K_r \square K_t) \geq \left\lceil \frac{rt(r+t-1)}{2(r+t)} \right\rceil$. The proof is complete. \square

Corollary 27. For any complete graphs

$$\gamma_o(K_r \square K_t) \geq \gamma_o(K_r) \gamma_o(K_t).$$

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References

- [1] G. Abay-Asmerom, R. Hammack, C.E. Larson, D.T. Taylor, Notes on the independence number in Cartesian product of graphs, *Discussiones Mathematicae Graph Theory* 31 (2011) 25–35.
- [2] S. Bermudo, J.A. Rodríguez-Velázquez, J.M. Sigarreta, I.G. Yero, On global offensive k -alliances in graphs, *Applied Mathematics Letters* 23 (2010) 1454–1458.
- [3] R.C. Brigham, R. Dutton, S. Hedetniemi, A sharp lower bound on the powerful alliance number of $C_m \times C_n$, *Congressus Numerantium* 167 (2004) 57–63.
- [4] O. Favaron, G. Fricke, W. Goddard, S.M. Hedetniemi, S.T. Hedetniemi, P. Kristiansen, R.C. Laskar, R.D. Skaggs, Offensive alliances in graphs, *Discussiones Mathematicae Graph Theory* 24 (2) (2004) 263–275.
- [5] J.P. Hayes, T.N. Mudge, Q.F. Stout, S. Colley, J. Palmer, Architecture of a hypercube supercomputer, in: *Proceeding of the International Conference on Parallel Processing*, 1986, pp. 653–660.
- [6] T.W. Haynes, S.T. Hedetniemi, P.J. Slater, *Fundamentals of Domination in Graphs*, Marcel Dekker, Inc., New York, 1998.
- [7] S.H. Hedetniemi, Homomorphisms of graphs and automata, University of Michigan Technical Report 03105-44-T, 1966.
- [8] W. Imrich, S. Klavžar, *Product Graphs, Structure and Recognition*, Wiley-Interscience, 2000.
- [9] W. Imrich, S. Klavžar, D. Rall, *Topics in Graph Theory: Graphs and their Cartesian Product*, A.K. Peters Ltd., Wellesley, MA, 2008.
- [10] P.C. Kainen, A lower bound for crossing numbers of graphs with applications to K_n , $K_{q,p}$, and Q_n , *Journal of Combinatorial Theory—Series B* 12 (1972) 287–289.
- [11] P. Kristiansen, S.M. Hedetniemi, S.T. Hedetniemi, Alliances in graphs, *Journal of Combinatorial Mathematics and Combinatorial Computing* 48 (2004) 157–177.
- [12] B. Mohar, Laplace eigenvalues of graphs—a survey, *Discrete Mathematics* 109 (1992) 171–183.
- [13] K.H. Shafique, R.D. Dutton, Maximum alliance-free and minimum alliance-cover sets, *Congressus Numerantium* 162 (2003) 139–146.
- [14] K.H. Shafique, R. Dutton, A tight bound on the cardinalities of maximum alliance-free and minimum alliance-cover sets, *Journal of Combinatorial Mathematics and Combinatorial Computing* 56 (2006) 139–145.
- [15] C.E. Shannon, The zero-error capacity of a noisy channel, *IRE Transactions on Information Theory* 2 (3) (1956) 8–19.
- [16] J.M. Sigarreta, J.A. Rodríguez-Velázquez, On the global offensive alliance number of a graph, *Discrete Applied Mathematics* 157 (2) (2009) 219–226.
- [17] J.M. Sigarreta, I.G. Yero, S. Bermudo, J.A. Rodríguez-Velázquez, Partitioning a graph into offensive k -alliances, *Discrete Applied Mathematics* 159 (2011) 224–231.
- [18] V.G. Vizing, The Cartesian product of graphs, *Vychislitelnye Sistemy* 9 (1963) 30–43.
- [19] V.G. Vizing, Some unsolved problems in graph theory, *Uspehi Matematicheskikh Nauk* 23 (144) (1968) 117–134.
- [20] I.G. Yero, S. Bermudo, J.A. Rodríguez-Velázquez, J.M. Sigarreta, Partitioning a graph into defensive k -alliances, *Acta Mathematica Sinica (English Series)* 27 (1) (2011) 73–82.